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# Compact embeddings of Brézis-Wainger type

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## Abstract

Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$  and denote by  $\text{id}_\Omega$  the restriction operator from the Besov space  $B_{pq}^{1+n/p}(\mathbb{R}^n)$  into the generalized Lipschitz space  $\text{Lip}^{(1,-\alpha)}(\Omega)$ . We study the sequence of entropy numbers of this operator and prove that, up to logarithmic factors, it behaves asymptotically like  $e_k(\text{id}_\Omega) \sim k^{-1/p}$  if  $\alpha > \max(1 + 2/p - 1/q, 1/p)$ . Our estimates improve previous results by Edmunds and Haroske.

## 1. Introduction

A famous result by Brézis and Wainger [2] states that every function  $f$  in the (fractional) Sobolev space  $H_p^{1+n/p}(\mathbb{R}^n)$ ,  $1 < p < \infty$ , is almost Lipschitz continuous in the sense that

$$|f(x) - f(y)| \leq c \|x - y\| |\log \|x - y\||^{1/p'} \|f\|_{H_p^{1+n/p}(\mathbb{R}^n)}$$

for all  $x, y \in \mathbb{R}^n$  with  $0 < \|x - y\| < 1/2$ , where the constant  $c$  is independent of  $x, y$ , and  $f$  and, as usual,  $p'$  is defined by  $1/p + 1/p' = 1$ .

Motivated by this result, Edmunds and Haroske [8, 9] introduced the Banach spaces of Lipschitz type  $\text{Lip}^{(1,-\alpha)}(\mathbb{R}^n)$ ,  $\alpha \geq 0$ , consisting of all complex valued, continuous functions on  $\mathbb{R}^n$  such that

$$\|f\|_{\text{Lip}^{(1,-\alpha)}(\mathbb{R}^n)} = \|f\|_{L_\infty(\mathbb{R}^n)} + \sup_{0 < \|x-y\| < 1/2} \frac{|f(x) - f(y)|}{\|x - y\| |\log \|x - y\||^\alpha}$$

is finite. Similarly, they introduced the spaces  $\text{Lip}^{(1,-\alpha)}(\Omega)$  for bounded domains  $\Omega$  in  $\mathbb{R}^n$ , and studied embeddings of Sobolev and Besov spaces into

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these Lipschitz spaces. In the context of Besov spaces they showed, in particular, that if  $0 < p, q \leq \infty$ , then there exists an embedding  $B_{pq}^{1+n/p}(\mathbb{R}^n) \hookrightarrow \text{Lip}^{(1,-\alpha)}(\mathbb{R}^n)$  if and only if  $\alpha \geq \max(1 - 1/q, 0)$ . If  $\mathbb{R}^n$  is replaced by a bounded open domain  $\Omega \subseteq \mathbb{R}^n$  with smooth boundary, then the corresponding embedding is compact if and only if  $\alpha > \max(1 - 1/q, 0)$ . Moreover, they gave two sided estimates for the entropy numbers of these compact embeddings, which are often called “limiting,” meaning that the differential dimensions of the domain and target space coincide. The differential dimension of  $B_{pq}^s(\mathbb{R}^n)$  is  $s - n/p$ , in our case it is equal to 1.

In a previous paper [7] the first two of the present authors improved the upper estimates of Edmunds and Haroske [8, 9] for the Banach space case, that is, for  $p, q \geq 1$ . Our aim here is to deal with the quasi-Banach space case, where  $p$  and/or  $q$  are less than 1. Some of the techniques used in [7] do not work in this case.

In this paper we establish first a new lower entropy estimate and then we improve the upper entropy estimates to the effect that, up to logarithmic factors, the exact asymptotic behavior of the entropy numbers of the embedding turns out to be like  $(k^{-1/p})$  if  $\alpha \geq 1 + 2/p - 1/q$  and  $1/q < 1 + 1/p$ , or if  $\alpha > 1/p$  and  $1/q \geq 1 + 1/p$ . The organization of the paper is as follows. In Section 2 we review some known facts on entropy numbers and function spaces, and state a few preliminary results. In Section 3 we establish an estimate for entropy numbers in sequence spaces that shall be used in the last section. Finally, the lower and upper entropy bounds for the embeddings in function spaces will be proved in Sections 4 and 5, respectively.

## 2. Preliminaries

In what follows, all the quasi-Banach spaces under consideration are defined over the field  $\mathbb{C}$  of complex numbers, except if otherwise noted. Let  $X, Y$  be quasi-Banach spaces and let  $T \in \mathcal{L}(X, Y)$  be a bounded linear operator from  $X$  to  $Y$ . For each  $k \in \mathbb{N}$ , the  $k$ -th (dyadic) entropy number  $e_k(T)$  is defined by

$$e_k(T) = \inf\{\varepsilon > 0 : T(B_X) \subseteq \bigcup_{j=1}^{2^{k-1}} (y_j + \varepsilon B_Y) \text{ for some } y_1, \dots, y_{2^{k-1}} \in Y\},$$

where  $B_X, B_Y$  denote the closed unit balls of the spaces  $X$  and  $Y$ , respectively (see [14] or [10]). Then clearly

$$\|T\| \geq e_1(T) \geq e_2(T) \geq \dots \geq 0,$$

and  $T$  is compact if and only if  $\lim_{k \rightarrow \infty} e_k(T) = 0$ . Hence, the asymptotic decay of the sequence  $(e_k(T))$  can be considered as a measure of the “degree of

compactness" of the operator  $T$ . Moreover, there is a close relation between entropy numbers and eigenvalues, which is the basis of many applications. If  $T \in \mathcal{L}(X, X)$  is compact, let  $(\lambda_k(T))$  be the sequence of eigenvalues of  $T$ , counted according to their multiplicities and ordered by decreasing modulus. Then, by the celebrated Carl-Triebel inequality (see [3], [5] and [10]),

$$|\lambda_k(T)| \leq \sqrt{2}e_k(T), \quad k \in \mathbb{N}.$$

Entropy numbers are multiplicative and additive, meaning that if  $X, Y, Z$  are quasi-Banach spaces and  $S, T \in \mathcal{L}(X, Y)$ ,  $R \in \mathcal{L}(Y, Z)$ , then

$$e_{k+m-1}(R \circ T) \leq e_k(R)e_m(T), \quad k, m \in \mathbb{N},$$

and, if  $Y$  is a Banach space, then

$$e_{k+m-1}(S + T) \leq e_k(S) + e_m(T), \quad k, m \in \mathbb{N}.$$

We shall frequently use these properties in the next sections.

Let  $\Omega \subseteq \mathbb{R}^n$  be a bounded domain. Let  $0 \leq \alpha_0 \leq \alpha_1$ ,  $0 < \theta < 1$  and put  $\alpha = (1 - \theta)\alpha_0 + \theta\alpha_1$ . It is easy to check that for any  $f \in \text{Lip}^{(1, -\alpha_0)}(\Omega)$  we have

$$\|f\|_{\text{Lip}^{(1, -\alpha)}(\Omega)} \leq \|f\|_{\text{Lip}^{(1, -\alpha_0)}(\Omega)}^{1-\theta} \cdot \|f\|_{\text{Lip}^{(1, -\alpha_1)}(\Omega)}^\theta,$$

whence, according to an interpolation property of the entropy numbers (see [14, Proposition 12.1.12], or [10, Theorem 1.3.2],) we obtain:

**Lemma 2.1** *Let  $X$  be a quasi-Banach space. Let  $0 \leq \alpha_0 \leq \alpha_1$ ,  $0 < \theta < 1$  and set  $\alpha = (1 - \theta)\alpha_0 + \theta\alpha_1$ . Assume  $T \in \mathcal{L}(X, \text{Lip}^{(1, -\alpha_0)}(\Omega))$ . Then, for all  $k, m \in \mathbb{N}$ ,*

$$\begin{aligned} e_{k+m-1} \left( T : X \rightarrow \text{Lip}^{(1, -\alpha)}(\Omega) \right) \\ \leq 2e_k \left( T : X \rightarrow \text{Lip}^{(1, -\alpha_0)}(\Omega) \right)^{1-\theta} e_m \left( T : X \rightarrow \text{Lip}^{(1, -\alpha_1)}(\Omega) \right)^\theta. \end{aligned}$$

To state another interpolation property of the entropy numbers, we first recall the definition of the real interpolation method. Let  $(X_0, X_1)$  be a compatible couple of quasi-Banach spaces, let  $0 < q \leq \infty$  and let  $0 < \theta < 1$ . The real interpolation space  $(X_0, X_1)_{\theta, q}$  consists of all  $x \in X_0 + X_1$  that have a finite quasi-norm

$$\|x\|_{(X_0, X_1)_{\theta, q}} = \left( \int_0^\infty (t^{-\theta} K(t, x))^q \frac{dt}{t} \right)^{1/q}$$

(modified as usual if  $q = \infty$ ), where  $K(t, x)$  is the  $K$ -functional of Peetre,

$$K(t, x) = \inf \{ \|x_0\|_{X_0} + t \|x_1\|_{X_1} : x = x_0 + x_1, x_j \in X_j \}$$

(see [1], [16]).

**Lemma 2.2** *Let  $(X_0, X_1)$  be an interpolation couple of quasi-Banach spaces and let  $Y$  be a Banach space. Let  $0 < \theta < 1$ ,  $0 < q \leq \infty$  and assume that  $T : X_0 + X_1 \rightarrow Y$  is a linear operator whose restrictions to  $X_0$  and  $X_1$  are continuous. Then the restriction of  $T$  to  $(X_0, X_1)_{\theta, q}$  is also continuous and we have for all  $k, m \in \mathbb{N}$ ,*

$$e_{k+m-1}(T : (X_0, X_1)_{\theta, q} \rightarrow Y) \leq 2e_k(T : X_0 \rightarrow Y)^{1-\theta}e_m(T : X_1 \rightarrow Y)^\theta.$$

For more details on entropy numbers we refer to the monographs by Pietsch [14], König [11], Carl and Stephani [4], Edmunds and Triebel [10], and Triebel [17].

We denote by  $\ell_p^N$  the space  $\mathbb{C}^N$  with the quasi-norm

$$\|x\|_{\ell_p^N} = \begin{cases} \left( \sum_{k=1}^N |x_k|^p \right)^{1/p} & \text{if } 0 < p < \infty, \\ \sup_{1 \leq k \leq N} |x_k| & \text{if } p = \infty. \end{cases}$$

Given a sequence of quasi-Banach spaces  $X_j$  and a sequence of positive real numbers  $w_j$ , indexed by  $j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ , we denote by  $\ell_p(w_j X_j)$ ,  $0 < p \leq \infty$ , the weighted vector-valued space consisting of all sequences  $x = (x_j)$  such that  $x_j \in X_j$  and

$$\|x\|_{\ell_p(w_j X_j)} = \left( \sum_{j=0}^{\infty} w_j^p \|x_j\|_{X_j}^p \right)^{1/p} < \infty$$

(with the usual modification if  $p = \infty$ ). If  $X_j = \mathbb{C}$  for all  $j$ , we denote this space by  $\ell_p(w_j)$ . If  $w_j = 1$  for all  $j$ , we write  $\ell_p(X_j)$ . The case in which  $X_j = \ell_q^{M_j}$  with  $0 < q \leq \infty$ ,  $M_j \in \mathbb{N}$  and  $w_j = (j+1)^\alpha$  for some  $\alpha > 0$ , will be of special interest to us.

We shall find the following result useful. It is an easy consequence of [16], Theorem 1.4.2, and Minkowski's inequality (see [6, Lemma 1]).

**Lemma 2.3** *Let  $(X_j, Y_j)$ ,  $j \in \mathbb{N}_0$ , be compatible couples of quasi-Banach spaces. Assume that  $0 < q_0, q_1 < \infty$ ,  $0 < \theta < 1$ ,  $1/q = (1-\theta)/q_0 + \theta/q_1$  and  $0 < q \leq p < \infty$ . Then  $\ell_q((X_j, Y_j)_{\theta, p})$  is continuously embedded in  $(\ell_{q_0}(X_j), \ell_{q_1}(Y_j))_{\theta, p}$ .*

Next we recall the classical Fourier analytical definition of the Besov spaces. Let  $\varphi$  be a  $C^\infty$  function on  $\mathbb{R}^n$  with  $\text{supp } \varphi \subseteq \{y \in \mathbb{R}^n : \|y\| \leq 2\}$  and  $\varphi(x) = 1$  if  $\|x\| \leq 1$ . Set  $\varphi_0 = \varphi$  and for each  $j \in \mathbb{N}$  set  $\varphi_j(x) = \varphi(2^{-j}x) - \varphi(2^{-j+1}x)$ .

If  $0 < p, q \leq \infty$ ,  $s \in \mathbb{R}$ , the Besov space  $B_{pq}^s(\mathbb{R}^n)$  is the space of all complex valued tempered distributions  $f$  satisfying

$$\|f\|_{B_{pq}^s(\mathbb{R}^n)} = \left( \sum_{j=0}^{\infty} 2^{jsq} \|\mathcal{F}^{-1} \varphi_j \mathcal{F} f\|_{L_p(\mathbb{R}^n)}^q \right)^{1/q} < \infty$$

(modified as usual if  $q = \infty$ ). Here  $\mathcal{F}, \mathcal{F}^{-1}$  stand for the Fourier transform and the inverse Fourier transform, respectively. There are many other equivalent descriptions of Besov spaces; we shall use the so-called subatomic decompositions, which are due to Triebel, see [17, Chapter 14].

Logarithms are taken mostly in base 2, only occasionally in base  $e$ ; we use the notation  $\log = \log_2$  and  $\ln = \log_e$ , respectively.

Given two sequences  $(a_k)$  and  $(b_k)$  of positive real numbers we write  $a_k \preceq b_k$  if there exists a constant  $c > 0$  such that  $a_k \leq cb_k$  for all  $k \in \mathbb{N}$ , and  $a_k \sim b_k$  if  $a_k \preceq b_k$  and  $b_k \preceq a_k$ . Moreover, since we shall often neglect logarithmic factors, we define  $a_k \stackrel{\log}{\prec} b_k$  to mean that there exist constants  $c > 0$  and  $\rho \in \mathbb{R}$  such that

$$a_k \leq c(1 + \log k)^\rho b_k \quad \text{for all } k \in \mathbb{N}.$$

If  $a_k \stackrel{\log}{\prec} b_k$  and  $b_k \stackrel{\log}{\prec} a_k$ , then we write  $a_k \stackrel{\log}{\sim} b_k$ .

### 3. Entropy numbers and sequence spaces

We start with another auxiliary result.

**Lemma 3.1** *Let  $0 < p < \infty$ . Then there is a constant  $\eta = \eta(p) > 0$  such that*

$$\sum_{j=1}^{\infty} r^{-jp} \leq \frac{\eta}{r} \quad \text{for all } r \geq 2.$$

**Proof.** For any  $r \geq 2$  we have

$$\begin{aligned} r \sum_{j=1}^{\infty} r^{-jp} &= 1 + \sum_{j=2}^{\infty} r^{1-jp} \leq 1 + \int_1^{\infty} r^{1-xp} dx = \\ &= 1 + \frac{1}{p} \int_0^{\infty} r^{-t} (1+t)^{1/p-1} dt \leq 1 + \frac{1}{p} \int_0^{\infty} 2^{-t} (1+t)^{1/p-1} dt < \infty. \end{aligned}$$

The lemma is proved. ■

Moreover we shall use the following result, which is due to Schütt [15] in the Banach space case. The extension to the quasi-Banach space case was given in [10, Proposition 3.2.2] and [12].

**Lemma 3.2** *Let  $0 < p < \infty$ , then (for real spaces)*

$$e_k(\text{id} : l_p^m \rightarrow l_\infty^m) \sim \begin{cases} 1 & \text{if } 1 \leq k \leq \log m \\ \left(\frac{\log(\frac{m}{k}+1)}{k}\right)^{1/p} & \text{if } \log m \leq k \leq m \\ 2^{-\frac{k-1}{m}} m^{-1/p} & \text{if } k \geq m. \end{cases}$$

*The formula remains valid for complex spaces, if on the right hand side  $m$  is replaced by  $2m$ .*

Next we establish an estimate for entropy numbers in sequence spaces which will be crucial for our later results in Section 5. It extends a result of [13] from the Banach space case to the case of quasi-Banach spaces.

**Theorem 3.3** *Let  $0 < p < \infty$  and let  $w_j = (\log(j+1))^{1/p}$  for  $j \in \mathbb{N}$ . Then the entropy numbers of the embedding  $\text{id} : \ell_p(w_j) \rightarrow \ell_\infty$  satisfy*

$$(3.1) \quad e_k(\text{id}) \sim k^{-1/p}.$$

**Proof.** We prove the result only for *real* sequence spaces, then the *complex* case follows easily by identifying  $\mathbb{C}$  with  $\mathbb{R}^2$ .

The lower estimate can be proved in the same way as Theorem 3 of [13], except that the Theorem in [12], which extends the lower estimates of [15] to the quasi-Banach case, needs to be used. Given  $m \in \mathbb{N}$ , let  $P_m : \ell_p(w_j) \rightarrow \ell_\infty$  be the projection onto the coordinates  $j = 1, \dots, m$ . Then the properties of entropy numbers yield

$$e_k(\text{id} : \ell_p(w_j) \rightarrow \ell_\infty) \geq e_k(P_m : \ell_p(w_j) \rightarrow \ell_\infty) \geq w_m^{-1/p} e_k(\text{id} : l_p^m \rightarrow l_\infty^m),$$

and choosing  $m = k^2$  we obtain from Lemma 3.2 the desired estimate

$$e_k(\text{id} : \ell_p(w_j) \rightarrow \ell_\infty) \succeq k^{-1/p}.$$

Now we turn to the upper estimate. For  $\lambda > 0$  and  $N \in \mathbb{N}$ , consider the set

$$S_N(\lambda) := \{x = (x_j) \in \mathbb{Z}^\mathbb{N} : \sum_{j=1}^N w_j^p |x_j|^p \leq \lambda, x_{N+1} = x_{N+2} = \dots = 0\}$$

and let

$$\sigma_N(\lambda) := \text{card } S_N(\lambda).$$

We claim that it suffices to prove that there exist constants  $c > 0$  and  $d \in \mathbb{N}$  such that

$$(3.2) \quad \sigma_N(\lambda) \leq cN^d 2^\lambda \quad \text{for all } N \in \mathbb{N}, \lambda > 0.$$

In fact, choosing  $N = 2^k$  and  $\lambda = k \in \mathbb{N}$ , inequality (3.2) implies

$$(3.3) \quad \sigma_{2^k}(k) \leq c2^{(d+1)k}.$$

Let  $y = (y_j) \in \ell_p(w_j)$  with  $\|y\|_{\ell_p(w_j)} \leq k^{1/p}$ . Define  $x = (x_j) \in \mathbb{Z}^{\mathbb{N}}$  by

$$x_j = \begin{cases} \operatorname{sgn} y_j \cdot [|y_j|] & \text{if } 1 \leq j \leq 2^k \\ 0 & \text{if } j > 2^k, \end{cases}$$

where  $[\cdot]$  is the greatest integer function. Then  $|x_j| \leq |y_j|$  for all  $j \in \mathbb{N}$ , whence  $x \in S_{2^k}(k)$ . In addition, we have  $|x_j - y_j| \leq 1$  for  $j = 1, \dots, 2^k$ , and

$$|x_j - y_j| = |y_j| \leq \frac{k^{1/p}}{w_j} \leq \frac{k^{1/p}}{w_{2^{k+1}}} \leq 1 \quad \text{for } j > 2^k.$$

Thus  $\|x - y\|_{\ell_\infty} \leq 1$ , showing that the balls in  $\ell_\infty$  of radius 1 centered at the points of  $S_{2^k}(k)$  cover  $k^{1/p}B_{\ell_p(w_j)}$ . By an obvious scaling argument, taking (3.3) into account, we obtain

$$e_{(d+1)k+k_0}(\operatorname{id} : \ell_p(w_j) \rightarrow \ell_\infty) \leq k^{-1/p}$$

where  $k_0 \in \mathbb{N}$  is chosen such that  $c \leq 2^{k_0-1}$ . This proves (3.1).

It remains to prove (3.2). To this end, we show that for each  $N \in \mathbb{N}$  there exists a constant  $\gamma_N$  such that

$$(3.4) \quad \sigma_N(\lambda) \leq \gamma_N 2^\lambda \quad \text{for all } \lambda > 0,$$

and determine the value of  $\gamma_N$ . We proceed by induction. Since  $\sigma_1(\lambda) = 1 + 2[\lambda^{1/p}]$ , we can take

$$\gamma_1 = \sup_{\lambda > 0} \frac{1 + 2\lambda^{1/p}}{2^\lambda} < \infty.$$

Let  $N \geq 2$  and assume  $\gamma_{N-1}$  has been determined. We see that  $x \in S_N(\lambda)$  if and only if  $x = (x', x_N, 0, 0, \dots)$ , where  $x' = (x_1, \dots, x_{N-1}) \in \mathbb{Z}^{N-1}$ ,  $x_N \in \mathbb{Z}$ ,  $|x_N| \leq r_N := \lambda^{1/p}/w_N$  and

$$(x', 0, 0, \dots) \in S_{N-1}(\lambda - w_N^p |x_N|^p).$$

Recalling that  $w_N^p = \log(N+1)$ , letting  $x_N$  range over all integers  $j$  with  $-r_N \leq j \leq r_N$ , we get the recursive formula

$$\begin{aligned} \sigma_N(\lambda) &= \sum_{|j| \leq r_N} \sigma_{N-1}(\lambda - |j|^p \log(N+1)) \\ &= \sigma_{N-1}(\lambda) + 2 \sum_{1 \leq j \leq r_N} \sigma_{N-1}(\lambda - j^p \log(N+1)). \end{aligned}$$

By the induction hypothesis, we obtain

$$\sigma_N(\lambda) \leq \gamma_{N-1} \left( 1 + 2 \sum_{1 \leq j \leq r_N} 2^{-j^p \log(N+1)} \right) 2^\lambda \leq \gamma_{N-1} \left( 1 + 2 \sum_{j=1}^{\infty} (N+1)^{-j^p} \right) 2^\lambda.$$

By Lemma 3.1 we have, with  $\eta = \eta(p)$ ,

$$\sum_{j=1}^{\infty} (N+1)^{-j^p} \leq \frac{\eta}{N+1} \quad \text{for all } N \in \mathbb{N}.$$

Consequently, we derived (3.4) with constant

$$\gamma_N = \left( 1 + \frac{2\eta}{N+1} \right) \gamma_{N-1}.$$

Finally, using the inequality  $1 + x \leq e^x$  for  $x \in \mathbb{R}$ , we get by induction

$$\gamma_N \leq \gamma_1 \prod_{j=2}^N e^{2\eta/(j+1)} = \gamma_1 e^{2\eta \sum_{j=2}^N 1/(j+1)} \leq \gamma_1 e^{2\eta \ln N} = \gamma_1 N^{2\eta}.$$

This establishes (3.2) with  $c = \gamma_1$ ,  $d = [2\eta] + 1$ , and completes the proof in the case of real sequence spaces.  $\blacksquare$

## 4. Lower estimates

Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$ . It is shown in [9], Theorem 2.1, that the restriction map  $\text{id}_\Omega : B_{pq}^{1+n/p}(\mathbb{R}^n) \rightarrow \text{Lip}^{(1,-\alpha)}(\Omega)$  defined by  $\text{id}_\Omega(f) = f|_\Omega$ , is bounded if  $\alpha \geq \max(1 - 1/q, 0)$ . In this section we determine lower bounds for the entropy numbers of  $\text{id}_\Omega$ .

**Theorem 4.1** *Let  $0 < p < \infty$ ,  $0 < q \leq \infty$ , and  $\alpha \geq \max(1 - 1/q, 0)$ . Assume that  $\Omega \subseteq \mathbb{R}^n$  is a bounded domain with  $[-1, 1]^n \subseteq \Omega$ , and let*

$$\text{id}_\Omega : B_{pq}^{1+n/p}(\mathbb{R}^n) \rightarrow \text{Lip}^{(1,-\alpha)}(\Omega)$$

*be the restriction map described above. Then*

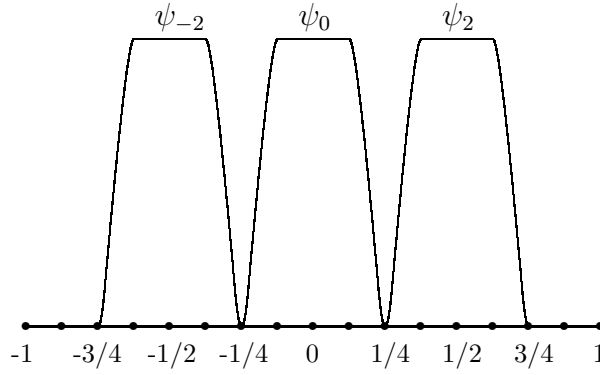
$$e_k(\text{id}_\Omega) \succeq \max \left( k^{-1/p} (\log k)^{1/p-\alpha}, k^{-\alpha} \right).$$



**Proof.** We shall use Triebel's subatomic decompositions of functions in the Besov space, referring to [17, Chapter III.14], for details. Let  $\psi \in C^\infty(\mathbb{R}^n)$  be such that  $\text{supp } \psi \subseteq Q := [-1, 1]^n$  and  $\psi(x) = 1$  if  $x \in [-1/2, 1/2]^n$ . Fix  $j \in \mathbb{N}$  and define for  $k \in (2\mathbb{Z})^n$ ,

$$\psi_k(x) = \psi(2^j x - k).$$

Let  $I$  be the set of multi-indices  $k \in (2\mathbb{Z})^n$  for which  $\text{supp } \psi_k \subseteq [-1, 1]^n$ . There are  $M_j := (2^j - 1)^n$  such functions  $\psi_k$ , and any two of them have non-overlapping supports. Figure 1 illustrates these functions for the case  $n = 1, j = 2$ .



**Figure 1**

Let  $A : \ell_p^{M_j} \rightarrow B_{pq}^{1+n/p}(\mathbb{R}^n)$  be the linear operator assigning to every  $\lambda = (\lambda_k)_{k \in I} \in \ell_p^{M_j}$  the function

$$A\lambda = 2^{-j} \sum_{k \in I} \lambda_k \psi_k.$$

Since  $Q \subseteq \Omega$ , we also can consider the linear operator  $B : \text{Lip}^{(1, -\alpha)}(\Omega) \rightarrow \ell_\infty^{M_j}$  defined by

$$Bf = 2^j (f(m_k) - f(r_k))_{k \in I},$$

where  $m_k = 2^{-j}k$  is the midpoint of the cube of side length  $2 \cdot 2^{-j}$  and  $r_k = 2^{-j}(1 + k_1, k_2, \dots, k_n)$ . According to [17, Chapter III.14], there is a constant  $c$  independent of  $j$  such that

$$\|A\lambda\|_{B_{pq}^{1+n/p}(\mathbb{R}^n)} = \left\| 2^{-j} \sum_{k \in I} \lambda_k \psi_k \right\|_{B_{pq}^{1+n/p}(\mathbb{R}^n)} \leq c \|\lambda\|_{\ell_p^{M_j}};$$

that is,  $\|A\| \leq c$ . On the other hand, since  $\|m_k - r_k\| = 2^{-j}$ , we have

$$\|Bf\|_{\ell_\infty^{M_j}} = 2^j \sup_{k \in I} |f(m_k) - f(r_k)| \leq 2^j \|f\|_{\text{Lip}^{(1, -\alpha)}(\Omega)} 2^{-j\alpha}$$

proving  $\|B\| \leq j^\alpha$ .

Since  $\psi_k(m_\ell) = \delta_{k\ell}$  and  $\psi_k(r_\ell) = 0$  for all  $k, \ell \in I$ , we can factorize the embedding  $\text{id} : \ell_p^{M_j} \rightarrow \ell_\infty^{M_j}$  as  $\text{id} = B \circ \text{id}_\Omega \circ A$ , obtaining the following commutative diagram

$$\begin{array}{ccc} \ell_p^{M_j} & \xrightarrow{\text{id}} & \ell_\infty^{M_j} \\ A \downarrow & & \uparrow B \\ B_{pq}^{1+n/p}(\mathbb{R}^n) & \xrightarrow{\text{id}_\Omega} & \text{Lip}^{(1,-\alpha)}(\Omega) \end{array}$$

Consequently,

$$e_m(\text{id}) \leq \|A\| e_m(\text{id}_\Omega) \|B\| \leq c j^\alpha e_m(\text{id}_\Omega).$$

Taking  $m = j$  (so  $\log M_j \sim j = m$ ) and using Lemma 3.2 we get

$$(4.1) \quad e_m(\text{id}_\Omega) \succeq m^{-\alpha}.$$

If instead we choose  $m = \sqrt{M_j}$  (thus  $\log m \sim \log M_j \sim j$ ), we get

$$(4.2) \quad e_m(\text{id}_\Omega) \succeq (\log m)^{1/p-\alpha} m^{-1/p}.$$

In conclusion,

$$\max(m^{-1/p}(\log m)^{1/p-\alpha}, m^{-\alpha}) \preceq e_m(\text{id}_\Omega).$$

■

**Remark.** The lower bound (4.2) improves the estimate

$$e_k(\text{id}_\Omega) \succeq k^{-1/p}(\log k)^{-\alpha},$$

which was established by Edmunds and Haroske via different proofs (see [8, Theorem 4.10] and [9, Theorem 3.11]).

**Remark.** The lower bound (4.1) is new. It shows, in particular, that if  $0 < q \leq 1$ , the map

$$\text{id}_\Omega : B_{pq}^{1+n/p}(\mathbb{R}^n) \rightarrow \text{Lip}^{(1,0)}(\Omega)$$

is not compact because  $\lim_{k \rightarrow \infty} e_k(\text{id}_\Omega) > 0$ .

## 5. Upper estimates

The restriction operator

$$\text{id}_\Omega : B_{pq}^{1+n/p}(\mathbb{R}^n) \rightarrow \text{Lip}^{(1,-\alpha)}(\Omega)$$

is compact if and only if  $\alpha > \max(1 - 1/q, 0)$  (see [9, Proposition 2.5]). In this section we determine new upper bounds for the entropy numbers of  $\text{id}_\Omega$ , which improve earlier results in [9] and [7]. The main emphasis will be on the case of quasi-Banach spaces.

**Theorem 5.1** *Let  $0 < p < \infty$  and  $0 < q \leq \infty$  with  $1/q < 1 + 1/p$ . Set*

$$\alpha_0 = \alpha_0(q) = \max(1 - 1/q, 0) \quad \text{and} \quad \alpha_1 = \alpha_1(p, q) = 1 + 2/p - 1/q.$$

*Then, for any bounded domain  $\Omega$  in  $\mathbb{R}^n$ , the entropy numbers of the operator*

$$\text{id}_\Omega : B_{pq}^{1+n/p}(\mathbb{R}^n) \rightarrow \text{Lip}^{(1,-\alpha)}(\Omega)$$

*satisfy the upper estimates*

$$e_k(\text{id}_\Omega) \stackrel{\log}{\prec} \begin{cases} k^{-1/p} & \text{if } \alpha \geq \alpha_1, \\ k^{-\theta/p} & \text{if } \alpha_0 < \alpha < \alpha_1 \quad \text{and} \quad \theta = (\alpha - \alpha_0)/(\alpha_1 - \alpha_0). \end{cases}$$

**Proof.** The first step of the proof consists in reducing the problem to the setting of sequence spaces, which can be done in the same way as in [7, Theorem 3] (and the references cited there) via subatomic decompositions of functions in  $B_{pq}^{1+n/p}(\mathbb{R}^n)$  (see [17]); therefore we shall not repeat the arguments. Let us only point out that it suffices to prove the entropy estimates for the embedding

$$(5.1) \quad \text{id}_\alpha : \ell_q(\ell_p^{M_j}) \rightarrow \ell_r(\langle j \rangle^{1/r' - \alpha} \ell_\infty^{M_j})$$

instead of  $\text{id}_\Omega$ . Here  $\langle j \rangle = j + 1$  for  $j \in \mathbb{N}_0$ ,  $M_j \sim 2^{jn}$  and  $r$  is an arbitrary exponent in the range  $1 \leq r \leq \infty$ , which is at our disposal. In fact, in our later considerations, we shall only work with  $r = \infty$  or  $r = 1$ .

In view of Theorem 4.1, the best possible estimate for the entropy numbers of  $\text{id}_\alpha$  would be

$$(5.2) \quad e_k(\text{id}_\alpha) \stackrel{\log}{\prec} \max(k^{-1/p}, k^{-\alpha}).$$

The assumption on the parameters  $p, q$  yields that  $\alpha_1 \geq 1/p$ , so that for  $\alpha = \alpha_1$  we have  $\max(k^{-1/p}, k^{-\alpha}) = k^{-1/p}$ .

Our first objective will be to establish (5.2) for the case  $\alpha = \alpha_1$ . Then, of course, the optimal estimate (5.2) is valid for all  $\alpha > \alpha_1$ , due to the embedding

$$\text{Lip}^{(1, -\alpha_1)}(\Omega) \hookrightarrow \text{Lip}^{(1, -\alpha)}(\Omega).$$

Finally, to deal with the case  $\alpha \in (\alpha_0, \alpha_1)$ , we shall use interpolation in the Lipschitz spaces.

We proceed to prove (5.2) for  $\alpha = \alpha_1$ , distinguishing three cases depending on the values of  $p$  and  $q$ .

**Case 1.**  $q = p$ .

Taking  $r = \infty$  in (5.1), we have  $1/r' - \alpha_1 = -1/p$ , so let us consider the embedding

$$\text{id}_{\alpha_1} : \ell_p(\ell_p^{M_j}) \rightarrow \ell_\infty(\langle j \rangle^{-1/p} \ell_\infty^{M_j}).$$

Let

$$\sigma_{jk} = \langle j \rangle^{-1/p} \quad \text{for } j \in \mathbb{N}_0 \text{ and } 1 \leq k \leq M_j.$$

The non-increasing rearrangement of the doubly indexed sequence  $\sigma = (\sigma_{jk})$  satisfies

$$\sigma^*(m) \sim (\log(m+1))^{-1/p},$$

hence, setting

$$w_m := (\log(m+1))^{1/p},$$

we arrive at the desired estimate

$$e_k(\text{id}_{\alpha_1}) \sim e_k(\text{id} : \ell_p(w_m) \rightarrow \ell_\infty) \sim k^{-1/p},$$

where the last equivalence is due to Theorem 3.3.

**Case 2.**  $q > p$ .

We take again  $r = \infty$  so that the embedding (5.1) becomes

$$\text{id}_{\alpha_1} : \ell_q(\ell_p^{M_j}) \rightarrow \ell_\infty(\langle j \rangle^{1/q-2/p} \ell_\infty^{M_j}).$$

For  $N \in \mathbb{N}$  consider the projections  $P_N, Q_N$  defined on sequences  $x = (x_j)$  with  $x_j \in \ell_p^{M_j}$  by

$$P_N x = (x_0, \dots, x_{N-1}, 0, 0, \dots), \quad Q_N x = (0, \dots, 0, x_N, x_{N+1}, \dots).$$

Then

$$P_N + Q_N = \text{id}_{\alpha_1}.$$

The norm of  $Q_N$  as an operator from  $\ell_q(\ell_p^{M_j})$  to  $\ell_\infty(\langle j \rangle^{1/q-2/p} \ell_\infty^{M_j})$  can easily be estimated by

$$\|Q_N\| \leq (N+1)^{1/q-2/p} \leq N^{-1/p}.$$

To deal with  $P_N$ , we factorize it in the form  $P_N = \text{id}_2 \circ \text{id}_1$ , obtaining the commutative diagram

$$\begin{array}{ccc} \ell_q^N(\ell_p^{M_j}) & \xrightarrow{P_N} & \ell_\infty^N(\langle j \rangle^{1/q-2/p} \ell_\infty^{M_j}) \\ & \searrow \text{id}_1 \quad \nearrow \text{id}_2 & \\ & \ell_p^N(\langle j \rangle^{1/q-1/p} \ell_p^{M_j}) & \end{array}$$

Applying Hölder's inequality, we obtain that

$$\|\text{id}_1\| \leq c(\log N)^{1/p-1/q}$$

while, by Case 1, we have  $e_k(\text{id}_2) \preceq k^{-1/p}$ . Thus, the choice  $N = k$  yields

$$e_k(P_k) \leq \|\text{id}_1\| e_k(\text{id}_2) \preceq (\log k)^{1/p-1/q} k^{-1/p}.$$

Consequently,

$$e_k(\text{id}_{\alpha_1}) \leq e_k(P_k) + \|Q_k\| \preceq (\log k)^{1/p-1/q} k^{-1/p}.$$

**Case 3.**  $q < p$ .

We use interpolation to deal with this case. Taking now  $r = 1$  in (5.1), we start estimating the entropy numbers of the embedding

$$\text{id}_s : \ell_s^N(\ell_s^{M_j}) \rightarrow \ell_1^N(\langle j \rangle^{-1-1/s} \ell_\infty^{M_j}),$$

where  $0 < s < \infty$ . We use again the decomposition  $\text{id}_s = P_N + Q_N$ . This time we have

$$\|Q_N\| \preceq \begin{cases} N^{-1-1/s} & \text{if } s \leq 1, \\ N^{-2/s} & \text{if } 1 < s < \infty. \end{cases}$$

Factorizing  $P_N$  by the diagram

$$\begin{array}{ccc} \ell_s^N(\ell_s^{M_j}) & \xrightarrow{P_N} & \ell_1^N(\langle j \rangle^{-1-1/s} \ell_\infty^{M_j}) \\ & \searrow \text{id}_1 \quad \nearrow \text{id}_2 & \\ & \ell_\infty^N(\langle j \rangle^{-1/s} \ell_\infty^{M_j}) & \end{array}$$

we have, by Case 1,  $e_k(\text{id}_1) \preceq k^{-1/s}$ .

Clearly  $\|\text{id}_2\| \leq \log N$  so that taking  $N = k$  we get

$$(5.3) \quad e_k(\text{id}_s) \leq e_k(P_k) + \|Q_k\| \preceq k^{-1/s} \log k.$$

Estimate (5.3) will be one of the endpoints for our interpolation.

Let  $\theta = 1 + 1/p - 1/q$ , then  $0 < \theta < 1$ ; set  $s = \theta p$ . If  $X_0 = \ell_1(\ell_\infty^{M_j})$ ,  $X_1 = \ell_s(\langle j \rangle^{1+1/s} \ell_s^{M_j})$ , then  $(X_0, X_1)$  is a compatible couple of quasi-Banach spaces for which we can set up the following interpolation diagram:

$$\begin{array}{ccc} X_0 = \ell_1(\ell_\infty^{M_j}) & \xrightarrow{\text{id}_0} & \\ (X_0, X_1)_{\theta, p} & \xrightarrow{\text{id}_\theta} & \ell_1(\ell_\infty^{M_j}) \\ X_1 = \ell_s(\langle j \rangle^{1+1/s} \ell_s^{M_j}) & \xrightarrow{\text{id}_1} & \end{array}$$

Since

$$\frac{1}{q} = \frac{1-\theta}{1} + \frac{\theta}{s}, \quad \frac{1}{p} = \frac{1-\theta}{\infty} + \frac{\theta}{s}, \quad \alpha_1 = (1-\theta) \cdot 0 + \theta \cdot \left(1 + \frac{1}{s}\right),$$

it follows from Lemma 2.3 that  $\ell_q(\langle j \rangle^{\alpha_1} \ell_p^{M_j})$  is continuously embedded in  $(X_0, X_1)_{\theta, p}$ . In consequence, we obtain from Lemma 2.2 and from (5.3) that

$$\begin{aligned} e_k(\text{id}_{\alpha_1} : \ell_q(\ell_p^{M_j}) &\rightarrow \ell_1(\langle j \rangle^{-\alpha_1} \ell_\infty^{M_j})) = \\ &= e_k(\text{id}_\theta : \ell_q(\langle j \rangle^{\alpha_1} \ell_p^{M_j}) \rightarrow \ell_1(\ell_\infty^{M_j})) \\ &\leq c \|\text{id}_0\|^{1-\theta} e_k(\text{id}_1)^\theta \\ &= c(\log k)^\theta k^{-\theta/s} \\ &= c(\log k)^{1+1/p-1/q} k^{-1/p}. \end{aligned}$$

To complete the proof of the theorem, it remains to establish the case  $\alpha_0 < \alpha < \alpha_1$ . Let  $\theta \in (0, 1)$  be such that  $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$  and consider the interpolation diagram

$$\begin{array}{ccc} B_{pq}^{1+n/p}(\mathbb{R}^n) & \begin{array}{l} \xrightarrow{\text{id}_{\Omega,0}} \\ \xrightarrow{\text{id}_\Omega} \\ \xrightarrow{\text{id}_{\Omega,1}} \end{array} & \begin{array}{l} \text{Lip}^{(1,-\alpha_0)}(\Omega) \\ \text{Lip}^{(1,-\alpha)}(\Omega) \\ \text{Lip}^{(1,-\alpha_1)}(\Omega) \end{array} \end{array}$$

Applying the estimates established in cases 1 to 3 for  $\alpha_1$ , and using Lemma 2.1, we obtain

$$e_k \left( \text{id}_\Omega : B_{pq}^{1+n/p}(\mathbb{R}^n) \rightarrow \text{Lip}^{(1,-\alpha)}(\Omega) \right) \leq 2 \|\text{id}_{\Omega,0}\|^{1-\theta} e_k(\text{id}_{\Omega,1})^\theta \stackrel{\log}{\prec} k^{-\theta/p}.$$

Since  $\theta = (\alpha - \alpha_0)/(\alpha_1 - \alpha_0)$ , this finishes the proof.  $\blacksquare$

We want to single out two special cases of the theorem. If  $0 < p = q < 1$ , then  $\alpha_0 = 0$ ,  $\alpha_1 = 1 + 1/p$ , and we have  $\theta/p = \alpha/(1+p)$ . Writing down the result in this case, we get the following improvement of the upper estimate obtained by Edmunds and Haroske in [9, Theorem 3.11].

**Corollary 5.2** *Let  $0 < p = q < 1$  and let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$ . Then the entropy numbers of the map*

$$\text{id}_\Omega : B_{pp}^{1+n/p}(\mathbb{R}^n) \rightarrow \text{Lip}^{(1,-\alpha)}(\Omega)$$

*satisfy*

$$e_k(\text{id}_\Omega) \stackrel{\log}{\prec} \begin{cases} k^{-1/p} & \text{if } \alpha \geq 1 + 1/p, \\ k^{-\alpha/(1+p)} & \text{if } 0 < \alpha < 1 + 1/p. \end{cases}$$

Suppose now that  $q \geq 1$ . Then the additional condition  $1/q < 1 + 1/p$  is satisfied for all  $p \in (0, \infty)$ , and we have  $\theta/p = (\alpha - 1/q')/2$ . Thus we get the following result, which extends Theorem 3 of [7] to the range  $0 < p < 1$ .

**Corollary 5.3** *Let  $1 \leq q \leq \infty$ ,  $0 < p < \infty$  and let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$ . Then the entropy numbers of*

$$\text{id}_\Omega : B_{pq}^{1+n/p}(\mathbb{R}^n) \rightarrow \text{Lip}^{(1,-\alpha)}(\Omega)$$

*satisfy*

$$e_k(\text{id}_\Omega) \stackrel{\log}{\prec} \begin{cases} k^{-1/p} & \text{if } \alpha \geq 2/p + 1/q', \\ k^{-(\alpha-1/q')/2} & \text{if } 1/q' < \alpha < 2/p + 1/q'. \end{cases}$$

**Remark.** If we take into consideration also the logarithmic factors which are hidden in Corollary 5.3, it turns out that the proof of Theorem 5.1 yields, in the case  $q \geq p$  and  $\alpha \geq 1 + 2/p - 1/q$ ,

$$e_k(\text{id}_\Omega) \preceq k^{-1/p} (\log k)^{1/p-1/q},$$

which improves the estimate

$$e_k(\text{id}_\Omega) \preceq k^{-1/p} (\log k)^{1/p}$$

obtained by Cobos and Kühn in [7, Theorem 3].

Next we turn our attention to the case that has not yet been covered.

**Theorem 5.4** *Let  $0 < q < p < \infty$  with  $1/q \geq 1 + 1/p$ , and let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$ . Then the entropy numbers of the map*

$$\text{id}_\Omega : B_{pq}^{1+n/p}(\mathbb{R}^n) \rightarrow \text{Lip}^{(1,-\alpha)}(\Omega)$$

*satisfy*

$$e_k(\text{id}_\Omega) \stackrel{\log}{\prec} \begin{cases} k^{-1/p} & \text{if } \alpha > 1/p \\ k^{-\alpha/(1+\varepsilon)} & \text{if } 0 < \alpha \leq 1/p, \end{cases}$$

*where  $\varepsilon > 0$  is arbitrary.*

**Proof.** Assume first  $\alpha > 1/p$ . Then

$$1 + \frac{2}{p} - \alpha < 1 + \frac{1}{p},$$

whence there exists a number  $r \in (0, \infty)$  with

$$1 + \frac{2}{p} - \alpha < \frac{1}{r} < 1 + \frac{1}{p}.$$

On one hand, this implies  $r > q$ , thus there is an embedding  $B_{pq}^{1+n/p}(\mathbb{R}^n) \hookrightarrow B_{pr}^{1+n/p}(\mathbb{R}^n)$ . On the other hand, since  $\alpha > \alpha_1(p, r) = 1 + 2/p - 1/r$ , Theorem 5.1 applies, and by the multiplicativity of entropy numbers we obtain the desired estimate

$$\begin{aligned} e_k(\text{id}_\Omega : B_{pq}^{1+n/p}(\mathbb{R}^n) \rightarrow \text{Lip}^{(1,-\alpha)}(\Omega)) &\preceq \\ &\preceq e_k(\text{id}_\Omega : B_{pr}^{1+n/p}(\mathbb{R}^n) \rightarrow \text{Lip}^{(1,-\alpha)}(\Omega)) \stackrel{\log}{\prec} k^{-1/p}. \end{aligned}$$

Suppose now  $0 < \alpha \leq 1/p$ . Given any  $\varepsilon > 0$ , set  $\beta = (1 + \varepsilon)/p$  and  $\theta = \alpha/\beta$ ; obviously  $\theta \in (0, 1)$ . Now we use the interpolation property of the entropy numbers stated in Lemma 2.1. From the diagram

$$\begin{array}{ccc} & & \text{Lip}^{(1,0)}(\Omega) \\ & \nearrow \text{id}_{\Omega,0} & \\ B_{pq}^{1+n/p}(\mathbb{R}^n) & \xrightarrow{\text{id}_\Omega} & \text{Lip}^{(1,-\alpha)}(\Omega) \\ & \searrow \text{id}_{\Omega,1} & \\ & & \text{Lip}^{(1,-\beta)}(\Omega) \end{array}$$

and the estimate we just established (note that  $\beta > 1/p$ ), we derive

$$e_k(\text{id}_\Omega) \leq 2 \|\text{id}_{\Omega,0}\|^{1-\theta} e_k(\text{id}_{\Omega,1})^\theta \preceq e_k(\text{id}_{\Omega,1})^\theta \stackrel{\log}{\prec} k^{-\theta/p} = k^{-\alpha/(1+\varepsilon)}.$$

The proof is complete. ■



Using Theorems 4.1, 5.1, and 5.4, we can determine, up to logarithmic factors, the exact asymptotic behavior of the entropy numbers of  $\text{id}_\Omega$  for a large range of parameters.

**Corollary 5.5** *Let  $0 < p < \infty$ ,  $0 < q \leq \infty$ , and let  $\Omega \subseteq \mathbb{R}^n$  be a bounded open domain. Then the entropy numbers of the restriction operator  $\text{id}_\Omega : B_{pq}^{1+n/p}(\mathbb{R}^n) \rightarrow \text{Lip}^{(1,-\alpha)}(\Omega)$  satisfy*

$$e_k(\text{id}_\Omega) \stackrel{\log}{\asymp} k^{-1/p}$$

*in the following cases*

- (a)  $1/q < 1 + 1/p$  and  $\alpha \geq 1 + 2/p - 1/q$ ;
- (b)  $1/q \geq 1 + 1/p$  and  $\alpha > 1/p$ .

*Moreover, if  $1/q \geq 1 + 1/p$  and  $0 < \alpha \leq 1/p$ , then the almost sharp estimate*

$$k^{-\alpha} \stackrel{\log}{\prec} e_k(\text{id}_\Omega) \stackrel{\log}{\prec} k^{-\alpha/(1+\varepsilon)}$$

*holds, where  $\varepsilon > 0$  is arbitrary.*

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